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Finite Groups Which May Be Defined by Two Operators Satisfying Two Conditions.

By G. A. Miller.

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§ 1. *Introduction.*

In the present article the expression, "condition satisfied by two operators (s_1, s_2)," will be used exclusively for a relation which may be represented in the form $s_1^\alpha s_2^\beta s_1^\gamma s_2^\delta \ldots. = 1$, where $\alpha, \beta, \gamma, \delta, \ldots.$ are positive or negative integers. Unless the contrary is explicitly stated, the operators s_1, s_2 are supposed to be distinct and neither of them is the identity. It is clear that a single condition between these operators is insufficient to define a finite group; for, if s_1, s_2 satisfy the equation $s_1^\alpha s_2^\beta s_1^\gamma s_2^\delta \ldots. = 1$ but are not restricted in any other way, we may replace s_1, s_2 respectively by two commutative operators t_1, t_2 which have been so chosen that $t_1^{\alpha+\gamma+\ldots.}$ is the inverse of $t_2^{\beta+\delta+\ldots.}$. As this condition does not fix an upper limit for the orders of t_1, t_2, it results that the order of the group (G) generated by s_1, s_2 can not be limited by a single condition imposed upon s_1, s_2.

Since it is necessary to have at least two conditions between two or more operators to limit the order of the group or groups which may be generated by them, it is of interest to inquire what groups may be defined by the least possible number of conditions between two operators. Such a study appears especially useful in the application of group properties to other subjects.

The two conditions can not be of the form $s_1^\alpha = 1$, $s_2^\beta = 1$, since the order of the product of two operators is not limited by the orders of these operators. If the conditions are of the form

$$s_1^n = 1,^* \qquad s_1^\alpha = s_2^\beta,$$

the order of G has an upper limit when n and α are relatively prime, or when

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* A defining equation of the form $s_1^n = 1$ implies that the order of s_1 is exactly n.

$\beta = 1$, and only then. When one of these conditions is satisfied, G is a cyclic group and its order divides βn. Moreover, every cyclic group can clearly be generated in this way. For instance, the cyclic group of order 40 is completely defined by the conditions $s_1^{10} = 1$, $s_1^3 = s_2^4$. On the contrary, the two conditions $s_1^{10} = 1$, $s_1^4 = s_2^3$ are satisfied by two operators which generate a group whose order exceeds any given number, which is in accord with the general theorem stated above.

It may happen that a pair of conditions which does not impose an upper limit on the order of G is nevertheless very useful in exhibiting properties of G. While such pairs of conditions do not lie within the scope of the present paper, it seems desirable to consider briefly one such pair, in view of its usefulness in what follows. This pair is

$$s_1^\alpha = s_2^\beta, \qquad s_1 s_2 = s_2 s_1.$$

As s_1, s_2 are commutative, G is an abelian group, and the following theorem is evident: *If two commutative operators satisfy the condition* $s_1^\alpha = s_2^\beta$, *they generate the direct product of two cyclic groups whose orders are respectively the lowest common multiple of the orders of these operators, and a divisor of the highest common factor of* α, β. *The latter group may be the identity, and it must be the identity whenever* α, β *are relatively prime.*

Two operators which have a common square are known to generate a very elementary category of solvable groups.* By adding to this condition the order of their product there result the pair of conditions

$$(s_1 s_2)^n = 1, \qquad s_1^2 = s_2^2.$$

These are satisfied by the generators of only a finite number of groups for a given value of n, as may be seen from the following developments:

$$(s_1 s_2^{-1})^n = (s_1 s_2 \,.\, s_2^{-2})^n = (s_1 s_2)^n \,.\, s_2^{-2n} = s_2^{-2n},$$

since s_2^2 is invariant under G. As s_2^{-2n} is both invariant under G and also transformed into its inverse by each of the operators s_1, s_2, it results that the order of s_2 divides $4n$, and that the order of $s_1 s_2^{-1}$ divides $2n$. If these operators have the highest possible orders for a given value of n, the cyclic groups which they generate have two common operators; and hence it results that the order of G

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* *Archiv der Mathematik und Physik*, Vol. IX (1905), p. 6.

is a divisor of $4n^2$. An interesting illustrative example is furnished by the non-abelian group of order 16 which contains two cyclic subgroups of order 8.*

$$s_1 = ahgfedcb\,.\,iponmlkj, \qquad s_2 = ajgpencl\,.\,bohmfkdi,$$
$$s_1 s_2 = am\,.\,bj\,.\,co\,.\,dl\,.\,ei\,.\,fn\,.\,gk\,.\,hp, \qquad s_1 s_2^{-1} = aoek\,.\,blfp\,.\,cigm\,.\,dnhj.$$

In this example $n = 2$ and the order of G is exactly $4n^2$. On the other hand, the two conditions

$$(s_1 s_2)^2 = 1, \qquad s_1^2 = s_2^2$$

can not be satisfied by the two generators of another non-abelian group. If G is abelian, it is either the four group or the group of order 8 and of type (2, 1). Hence the theorem: *If the two generators of a group satisfy the conditions* $(s_1 s_2)^2 = 1$, $s_1^2 = s_2^2$, *the group is one of the following three: the four group, the abelian group of order* 8 *and of type* (2, 1), *the non-abelian group of order* 16 *which contains two cyclic subgroups of order* 8.

In a similar manner all the possible groups may be determined for any given value of n. Since the second condition is sufficient to restrict G to a very elementary category of solvable groups, it seems of little value to determine all the possible groups for special values of n. The theorem that the number of these groups corresponding to a given value of n must be finite is, however, useful. A much more interesting set of conditions is

$$s_1^n = 1, \qquad s_1 s_2 = s_2^\alpha s_1^\beta.$$

When either α or $\beta = 0$, or when $\alpha = 1 = \beta$, these conditions reduce to those which were considered above. These special cases will be excluded in what follows, and we shall therefore confine ourselves to non-abelian groups in considering the groups whose generators satisfy such conditions. A large number of interesting cases arise when either α or $\beta = \pm 1$. These cases will be completely investigated, as regards the number of possible groups, in what follows. A number of these cases were considered recently under the heading "Generalization of the Groups of Genus Zero,"† and the present article completes this generalization along certain lines, especially when $\beta = -1$.

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* *Quarterly Journal of Mathematics*, Vol. XXVIII (1896), p. 269.

† *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 1.

§ 2. *Conditions of the Form* $s_1^n = 1$, $s_1 s_2 = s_2^\alpha s_1^\beta$, *when* $\alpha = \pm 1$ *or when* $\beta = 1$.

As observed above, it may be assumed that neither of the two numbers α, β is 0, and that they are not both equal to unity. If both of them are equal to -1, the second condition implies only that the order of $s_1 s_2$ is 2; and hence it does not limit the order of s_2. That is, the two conditions $s_1^n = 1$, $s_1 s_2 = s_2^{-1} s_1^{-1}$ are not sufficient to fix an upper limit for the order of G, and hence do not come within the scope of the present paper. In what follows we may therefore assume that α and β are not both equal to -1.

With the restrictions noted in the preceding paragraph, we proceed to determine all the possible groups when $\beta = 1$. Since $s_1 s_2 s_1^{-1} = s_2^\alpha$, it results that $s_1^n s_2 s_1^{-n} = s_2 = s_2^{\alpha^n}$. That is, the order of s_2 divides $\alpha^n - 1$ and is completely determined when this number is a prime, and only then. In this case G is completely defined by the given conditions, and in all other cases they restrict G to a limited number of groups. For instance, when $s_1^5 = 1$, $s_1 s_2 = s_2^2 s_1$, it results that s_2 is of order 31 and that G is the subgroup of order 155 in the holomorph of the group of order 31. On the other hand, the conditions $s_1^6 = 1$, $s_1 s_2 = s_2^2 s_1$ are satisfied by two generators of the group of order 378 obtained by extending the cyclic group of order 63 by means of an operator of order 6 transforming a generator of this cyclic group into its square, and also by two generators of each one of its four subgroups of orders 18, 42, 54 and 126 respectively. When $\alpha = 1$, the order of G has no upper limit; for, if s_1, s_2 satisfy the given conditions s_1, $t_2 s_2$ (t_2 being any operator which is commutative with s_2) must also satisfy these conditions, since the second condition takes the form $s_2^{-1} s_1 s_2 = s_1^\beta$. Hence the theorem: *The conditions* $s_1^n = 1$, $s_1 s_2 = s_2^\alpha s_1^\beta$ *are satisfied by the two generators of only a finite number of groups when* $\beta = 1$ *and* $\alpha \neq 1$, *but they are satisfied by two generators of each one of an infinite system of distinct groups when* $\alpha = 1$.

Having considered all the possible cases when at least one of the two numbers α, β is unity, we may assume, in what follows, that neither of these exponents is unity. When $\alpha = -1$, the conditions assume the form

$$s_1^n = 1, \qquad s_1 s_2 = s_2^{-1} s_1^\beta.$$

From the latter it results that $s_1 s_2 = s_2^{-1} s_1^{-1} . s_1^{\beta+1}$, or $(s_1 s_2)^2 = s_1^{\beta+1}$. Since $s_1^{\beta+1}$ is a power of $s_1 s_2$ and commutative with s_1, it must also be commutative with s_2; and hence it is invariant under G. As it is assumed that $\beta \neq -1$, it results that *G contains at least one invariant operator besides the identity whenever* $\alpha = -1$. When n and $\beta + 1$ are relatively prime, G must be abelian, and hence the second condition becomes $s_2^2 = s_1^{\beta-1}$. From this it results that the order of G is either n or $2n$ and that G is also cyclic whenever n and $\beta + 1$ are relatively prime. When n is even, $\beta + 1$ and $\beta - 1$ are odd, and hence G is of order $2n$ in this case. That is, *G is completely determined by the two conditions $s_1^n = 1$, $s_1 s_2 = s_2^{-1} s_1^\beta$ whenever n is even and also n and $\beta + 1$ are relatively prime. If these conditions are satisfied, G is the cyclic group of order $2n$.* For instance, G is the cyclic group of order 24 when $s_1^{12} = 1$ and $s_1 s_2 = s_2^{-1} s_1^6$, or when $s_1^{12} = 1$ and $s_1 s_2 = s_2^{-1} s_1^4$; but when $s_1^9 = 1$, $s_1 s_2 = s_2^{-1} s_1^4$, G is either the cyclic group of order 9 or the cyclic group of order 18.

When n and $\beta + 1$ are not relatively prime, the order of G has no upper limit. This fact results from the following considerations: Since G is generated by $s_1 s_2$, s_1, the second of the given conditions may be replaced by $(s_1 s_2)^2 = s_1^{\beta+1}$. Hence it results that if $s_1 s_2$, s_1 satisfy these conditions, they are also satisfied by $t s_1 s_2$, $t_1 s_1$ (where t, t_1 are commutative with $s_1 s_2$, s_1); and t is of order 2, while the order of t_1 is a common factor of n and $\beta + 1$. As t, t_1 may be so selected that they satisfy these conditions and that also the order of $t t_1$ is an arbitrary number, the theorem under consideration has been proved. It has therefore been established that the conditions $s_1^n = 1$, $s_1 s_2 = s_2^{-1} s_1^\beta$ can determine a finite group only when n and $\beta + 1$ are relatively prime. If, in addition to this, n is an even number, G is the cyclic group of order $2n$, while it is either this cyclic group or the cyclic group of order n when n is an odd number.

§ 3. *General Considerations as Regards Conditions of the Form $s_1^n = 1$, $s_1 s_2 = s_2^\alpha s_1^{-1}$.*

These conditions may be represented as follows:

$$s_1^n = 1, \qquad (s_1 s_2)^2 = s_2^{\alpha+1}.$$

Hence $s_2^{\alpha+1}$ is invariant under G and $(s_1 s_2)^2 = (s_2 s_1)^2 = s_2^{\alpha+1}$. Moreover, if s_1, s_2 satisfy these conditions and if t_1, t_2 are commutative with each of the operators s_1, s_2, and satisfy the additional condition $t_1^n = t_2^{\alpha+1} = 1$, $(t_1 t_2)^2 = 1$, it is possible to replace s_1, s_2 respectively by $t_1 s_1$, $t_2 s_2$ in the given equations. Unless n and α are such as to restrict the group $\{t_1, t_2\}$ generated by t_1, t_2 to a group of finite order, it is possible to select t_1, t_2 so as to satisfy these conditions and also to generate a group whose order exceeds any given number. Hence the conditions $s_1^n = 1$, $s_1 s_2 = s_2^\alpha s_1^{-1}$ can not fix an upper limit for the order of G except possibly for a few small values of n and α. That is, in order that these conditions come within the scope of the present paper it is necessary that n, $\alpha + 1$ have one of the following pairs of values: $2, k$; $3, \pm 3$; $3, \pm 4$; $3, \pm 5$; $4, \pm 3$; $5, \pm 3$; $3, -2$; $4, -2$; $5, -2$.

The first of these pairs does not require consideration, as the second condition would become $s_2^\alpha s_1$, since $s_1 = s_1^{-1}$ when s_1 is of order 2. We also excluded the cases when $\alpha = -2$, since these conditions would imply that s_1, s_2 are commutative and hence the resulting conditions could be written in a form which was completely considered in § 1. It will be found that each pair of conditions corresponding to the given separate sets of values of n, $\alpha + 1$ will give rise to a small number of groups of comparatively low order. These groups are closely related to the groups of genus zero. In fact, they have these groups for quotient groups, so that many of their fundamental properties may be directly deduced from those of the groups of genus zero.

§ 4. *Groups Defined by the Conditions* $s_1^3 = 1$, $s_1 s_2 = s_2^\alpha s_1^{-1}$.

The following pairs of conditions require consideration when $n = 3$:

$$s_1^3 = 1,\ s_1 s_2 = s_2^2 s_1^{-1};\ s_1^3 = 1,\ s_1 s_2 = s_2^3 s_1^{-1};\ s_1^3 = 1,\ s_1 s_2 = s_2^4 s_1^{-1};\ s_1^3 = 1,\ s_1 s_2 = s_2^{-3} s_1^{-1};$$
$$s_1^3 = 1,\ s_1 s_2 = s_2^{-4} s_1^{-1};\ s_1^3 = 1,\ s_1 s_2 = s_2^{-5} s_1^{-1};\ s_1^3 = 1,\ s_1 s_2 = s_2^{-6} s_1^{-1}.$$

The first set of conditions may be expressed as follows:

$$s_1^3 = 1, \qquad (s_1 s_2)^2 = s_2^3.$$

These conditions have been considered in an equivalent but slightly different form,* and it has been proved that *there are just two non-abelian groups which may be generated by two operators satisfying the conditions* $s_1^3 = 1$, $s_1 s_2 = s_2^2 s_1^{-1}$. *These are the tetrahedral group and the group of order* 24 *which does not involve a subgroup of order* 12. *When* s_1, s_2 *are commutative, the order of* G *is* 3.

The second set of conditions is

$$s_1^3 = 1, \qquad (s_1 s_2)^2 = s_2^4.$$

To find an upper limit for the order of s_2 we may proceed as follows: Since $s_1 s_2$ and $s_2 s_1$ have a common square, the product of one of these operators into the inverse of the other is transformed into its inverse by each one of them. By taking such a power of this product as to arrive at an invariant operator, this operator is either the identity or it is of order 2. In the present case we have

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* *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 5.

$$(s_1 s_2 s_1^{-1} s_2^{-1})^3 = (s_2^3 s_1 s_2^{-1})^3 = s_2^3 s_1 s_2^2 s_1 s_2^2 s_1 s_2^{-1} = s_2^6 s_1^{-1} s_2^4 s_1^{-1} s_2^4 s_1^{-1} s_2^{-2} = s_2^{12}.$$

Hence the order of s_2 must be a divisor of 24. When this order is 2, G is the symmetric group of order 6; when it is 3, G is abelian; and when it is 4, G is the symmetric group of order 24 according to a well-known definition of this symmetric group. If the order of s_2 is 6, it is clear that $\{s_1, s_2^3\} \equiv$ the symmetric group of order 6, since $(s_1 s_2^3)^2 = 1$; hence $\{s_2^4, s_1, s_2^3\} \equiv G$ is the direct product of this symmetric group and the group of order 3 when the order of s_2 is 6.

If the order of s_2 is 8, the given conditions may be written in the form

$$t_1^2 = t_2^4, \qquad (t_1 t_2)^3 = 1, \qquad t_2^8 = 1.$$

It is known* that these conditions define a group of order 48 involving the group of order 24 which does not contain a subgroup of order 12. When the order of s_2 is 12, it is easy to see that $\{s_2^{-3}, s_1 s_2^{-4}\} \equiv$ the symmetric group of order 24, since $(s_2^{-3})^4 = 1$, $(s_1 s_2^{-4})^3 = 1$, $(s_2^{-3} s_1 s_2^{-4})^2 = (s_2^{-7} s_1)^2 = (s_2^5 s_1)^2 = s_2^8 (s_2 s_1)^2 = 1.$ As G is generated by this symmetric group and s_2^4, it results that G is the direct product of the symmetric group of order 24 and the group of order 3 whenever the order of s_2 is 12. Finally, when the order of s_2 is 24, it results that $\{s_2^3, s_2^{-3} s_1 s_2^{-4}\} \equiv$ to the group of order 48 noted above; for

$$(s_2^{-3})^4 = (s_2^{-3} s_1 s_2^{-4})^2,$$

$$(s_2^{-6} s_1 s_2^{-4})^3 = (s_2^{-10} s_1)^3 = (s_2^{-2} s_1)^3 = s_2^{-2} s_1 s_2^{-2} s_1 s_2^{-2} s_1 = s_2 s_1^{-1} s_2^{-3} s_2^3 s_1^{-1} s_2^{-3} . s_2^3 s_1^{-1} . s_2^{-1} = 1.$$

By adjoining s_2^8 to this group of order 48 we clearly obtain G. As the group generated by s_2^8 is composed of invariant operators and has only the identity in common with $\{s_2^3, s_2^{-3} s_1 s_2^{-4}\}$, G is the direct product of the group of order 3 and the group of order 48 given above, whenever the order of s_2 is 24. Combining these results, we arrive at the theorem: *If two generators of a non-abelian group satisfy the conditions* $s_1^3 = 1$, $s_1 s_2 = s_2^3 s_1^{-1}$, *it must be one of the following six groups: the symmetric group of order* 6, *the symmetric group of order* 24, *a group of order* 48 *involving the group of order* 24 *which does not contain a subgroup of order* 12, *or the direct product of the group of order* 3 *and one of these three groups. When* G *is abelian, it is the cyclic group of order* 6 *or the group of order* 3. *In the last case the two operators* s_1, s_2 *can, however, not be distinct.*

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* *Loc. cit.*, p. 7.

The third one of the given sets of conditions may be represented as follows:

$$s_1^3 = 1, \qquad (s_1 s_2)^2 = s_2^5.$$

To find an upper limit for the order of s_2 we may proceed in the following manner:

$$s_1 s_2 s_1^{-1} s_2^{-1} = s_2^4 s_1^{-2} s_2^{-1} = s_2^4 s_1 s_2^{-1},$$
$$(s_2^4 s_1 s_2^{-1})^2 = s_2^4 s_1 s_2^3 s_1 s_2^{-1} = s_2^8 s_1^{-1} s_2^6 s_1^{-1} s_2^{-2} = s_2^{13} s_1 s_2^4 s_1 s_2^{-2},$$
$$(s_2^4 s_1 s_2^{-1})^3 = s_2^{13} s_1 s_2^4 s_1 s_2^2 s_1 s_2^{-1} = s_2^{22} s_1^{-1} s_2^2 s_1^{-1} s_2^5 s_1^{-1} s_2^{-2} = s_2^{27} s_1^{-1} s_2^2 s_1 s_2^{-2} = s_2^{32} s_1^{-1} s_2 s_1^{-1} s_2^{-3},$$
$$(s_2^4 s_1 s_2^{-1})^5 = s_2^{32} s_1^{-1} s_2 s_1^{-1} s_2^{-3} . s_2^{13} s_1 s_2^4 s_1 s_2^{-2} = s_2^{45}.$$

As s_2^{45} is invariant under G and is also transformed into its inverse by $s_1 s_2$, it results that the order of s_2 divides 90. We may also assume that it is a multiple of 5, since we may confine ourselves to the consideration of non-abelian groups.

When s_2 is of order 5, G is the icosahedral group according to a familiar definition of this group. When this order is 10, the conditions may be expressed as follows: $(s_1 s_2)^4 = 1$, $s_2^5 = (s_1 s_2)^2$, $s_1^3 = 1$. Letting $t_1 = s_1 s_2$, $t_2 = s_2^{-1}$, these conditions become $t_1^4 = 1$, $t_2^5 = t_1^2$, $(t_1 t_2)^3 = 1$. The latter conditions define a group known as G_{120},* and hence this is the group defined by $s_1^3 = 1$, $s_1 s_2 = s_2^4 s_1^{-1}$, when the order of s_2 is 10. When this order is 15, the icosahedral group is generated by s_1, s_2^6, since $(s_2^6)^5 = 1$, $s_1^3 = 1$, $(s_1 s_2^6)^2 = s_2^{10} (s_1 s_2)^2 = s_2^{15} = 1$. Hence it results that G is the direct product of the icosahedral group and the group of order 3, when s_2 is of order 15. If this order is 30, it is evident that G_{120} is generated by $s_1 s_2^6$, s_2^{-9}, since $(s_1 s_2^6)^4 = 1$, $(s_2^{-9})^5 = (s_1 s_2^6)^2$, $(s_1 s_2^{-3})^3 = s_1 s_2^{-3} s_1 s_2^{-3} s_1 s_2^{-3} = s_1 s_2 s_1^{-1} s_2^{-4} . s_2^4 s_1^{-1} s_2^{-4} = 1$. Hence G is the direct product of G_{120} and the group of order 3, when the order of s_2 is 30. The next possible value of this order is 45. When s_2 is of order 45, $\{s_2^9, s_1 s_2^{12}\} \equiv$ icosahedral group, since $(s_2^9)^5 = 1$, $(s_1 s_2^{21})^2 = (s_1 s_2)^2 s_2^{40} = 1$, $(s_1 s_2^{12})^3 = (s_1 s_2^{-3})^3 = 1$. In this case G is therefore the direct product of the cyclic group of order 9, generated by s_2^5, and the icosahedral group. Finally, when the order of s_2 is 90, G_{120} is generated by $s_1 s_2^{21}$, s_2^9, since $(s_1 s_2^{21})^4 = 1$, $(s_2^9)^5 = (s_1 s_2^{21})^2$, $(s_1 s_2^{30})^3 = 1$. Hence G is the direct product of the cyclic group of order 9 and G_{120} when s_2 is of order 90. These results give rise to the following theorem: *Two non-commutative operators satisfying the conditions $s_1^3 = 1$, $s_1 s_2 = s_2^4 s_1^{-1}$ must generate one of the following six groups: the icosahedral*

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* *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 10.

group, G_{120}, or the direct product of one of these groups and one of the two cyclic groups of orders 3 *and* 9 *respectively. Each of these groups is completely determined by adding the order of s_2 to the given conditions, and hence it is completely defined by three conditions. If two commutative operators satisfy the conditions $s_1^3 = 1$, $s_1 s_2 = s_2^4 s_1^{-1}$, they generate the cyclic group of order* 9.

The set of conditions $s_1^3 = 1$, $s_1 s_2 = s_2^{-2} s_1^{-1}$ implies that s_2 is invariant under G and hence G is abelian. As $(s_1 s_2)^2 = s_1^2 s_2^2 = s_2^{-1}$, it results that s_2^3 is of order 3, and hence *the conditions $s_1^3 = 1$, $s_1 s_2 = s_2^{-2} s_1^{-1}$ define the cyclic group of order* 9. The next set of conditions is equivalent to $s_1^3 = 1$, $(s_1 s_2)^2 = s_2^{-2}$. Since $s_1 s_2$ and s_2^{-1} have a common square, $s_1 s_2^2$ is transformed into its inverse by each of these operators. An upper limit for the order of s_2 results from $(s_1 s_2^2)^3 = s_2^6$, s_2^2 being invariant under G. As s_2^6 both is invariant under G and is also transformed into its inverse by $s_1 s_2$, its order can not exceed 2. From this it results that the order of s_2 divides 12 and that the order of $s_1 s_2^2$ divides 6. If both of these operators have the largest possible orders, the order of G is 36. The remaining properties of this group of order 36 may easily be deduced from the fact that $s_1 s_2$ may be represented by the following substitutions:

$$s_1 = ace \,.\, bdf \,.\, klm, \qquad s_2 = afedcb \,.\, ghij \,.\, km.$$

When s_2 is of order 6 and $s_1 s_2^2$ is of order 3, G is of order 18 and may be generated by

$$s_1 = ace \,.\, bdf \,.\, klm, \qquad s_2 = afedcb \,.\, km.$$

If $s_1 s_2^2$ is the identity when s_2 is of order 6, G is clearly the cyclic group of order 6, and if $s_1 s_2^2$ is of of order 2 when s_2 is of order 12, G is the cyclic group of order 12. When s_2 is of order 4, $s_1 s_2^2$ must be of order 6, and hence G is the associate-dihedral* group of order 12. When s_2 is of order 3, G is the group of order 3, and when s_2 is of order 2, G is the symmetric group of order 6. Uniting these results we arrive at the theorem: *There are exactly seven groups which may be generated by two operators satisfying the two conditions $s_1^3 = 1$, $s_1 s_2 = s_2^{-3} s_1^{-1}$; these are the group of order* 3, *the two groups of order* 6, *the cyclic group and the associate-dihedral group of order* 12, *a group of order* 18, *and one of order* 36. This theorem illustrates the general theorem of § 1 as regards groups generated by two operators having a common square.

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* AMERICAN JOURNAL OF MATHEMATICS, Vol. XXIX (1907), p. 291.

The remaining three sets of conditions are respectively equivalent to the following:

$$s_1^3 = 1,\ (s_1 s_2)^2 = (s_2^{-1})^3; \quad s_1^3 = 1,\ (s_1 s_2)^2 = (s_2^{-1})^4; \quad s_1^3 = 1,\ (s_1 s_2)^2 = (s_2^{-1})^5.$$

The possible groups that may be generated by two operators satisfying these conditions were determined in Vol. VIII of the *Transactions of the American Mathematical Society,* pp. 4, 6 and 11 respectively. Hence all the cases which may give rise to only a finite number of groups when $n = 3$ have been considered.

§ 5. *Groups Defined by the Conditions* $s_1^4 = 1,\ s_1 s_2 = s_2^a s_1^{-1}$.

When $n = 4$, the following three sets require consideration:

$$s_1^4 = 1,\ s_1 s_2 = s_2^2 s_1^{-1}; \quad s_1^4 = 1,\ s_1 s_2 = s_2^{-3} s_1^{-1}; \quad s_1^4 = 1,\ s_1 s_2 = s_2^{-4} s_1^{-1}.$$

The first set can be satisfied only when the order of s_2 divides 12, since

$$(s_1 s_2 s_1^{-1} s_2^{-1})^3 = (s_2^2 s_1^2 s_2^{-1})^3 = s_2^2 s_1^2 s_2 s_1^2 s_2 s_1^2 s_2^{-1} = s_2^2 s_1 . s_1 s_2 . s_1 . s_1 s_2 . s_1^2 s_2^{-1} = s_2^6 .$$

As the order of s_2 is a multiple of 3 when G is non-abelian, it is necessary to consider the cases which arise when the order of s_2 has one of the following values: 3, 6, 12. In the first case G is the symmetric group of order 24. When s_2 is of order 6, $\{s_1^2 s_2,\ s_2^2\} \equiv$ the tetrahedral group, since

$$(s_2^2)^3 = 1,\ (s_1^2 s_2^3)^2 = 1,\ (s_1^2 s_2)^3 = s_1^2 s_2 s_1^2 s_2 s_1^2 s_2 = s_1 . s_1 s_2 . s_1 . s_1 s_2 . s_1^2 s_2 = s_1 s_2^4 s_1 s_2 = s_2^6 = 1.$$

This group is invariant under G, since it involves

$$s_2^{-1} s_1^2 s_2^2 = s_2^5 s_1^2 s_2^2 = s_2^2 s_1^2 s_2^5 = s_2^2 . s_1^2 s_2^3 . s_2^2 ,$$

$$s_1^{-1} s_2^2 s_1 = s_1^2 . s_1 s_2 . s_2 s_1 = s_1^2 s_2^2 s_1^2 s_2^2 = s_2^3 s_1^2 . s_2^2 . s_1^2 s_2^3 . s_2^{-4}, \text{ and } s_1^{-1} . s_1^2 s_2 . s_1 = s_1 s_2 s_1 = s_2^2 .$$

By adjoining s_2^3 to this invariant tetrahedral group there results the direct product of the group of order 2 and the tetrahedral group. This direct product is also invariant under G, since s_2^3 is invariant under G. The group of order 24 just obtained involves s_2 and s_1^2. Hence G is the group of order 48 obtained by adjoining s_1 to this group of order 24. When the order of s_2 is 12, G involves the quaternion group as an invariant subgroup. This subgroup may be generated by $s_1^2 s_2^3$, $s_2^{-1} . s_1^2 s_2^3 . s_2 = s_2^{-1} s_1^2 s_2^4$. The fact that these two operators generate the quaternion group results from the equations

$$(s_1^2 s_2^3)^4 = 1, \quad (s_1^2 s_2^3)^2 = (s_2^{-1} s_1^2 s_2^4)^2,$$

$$s_1^2 . s_2^{-1} s_1^2 s_2^4 . s_1^2 = s_2^3 s_1^2 s_2^{-1} s_1^2 s_2 s_1^2 = s_2^3 s_1^2 s_2^{-1} s_1 s_2^2 s_1 = s_2^3 s_1^2 s_2 s_1^{-1} . s_2 s_1 = s_2^3 s_1^2 s_2 s_1^{-2} s_2^2$$

$$= s_2^3 s_1 s_2^2 s_1 s_2^2 = s_2^5 s_1^{-1} s_2^3 s_1^{-1} s_2 = s_2^8 s_1^2 s_2 = (s_2^{-1} s_1^2 s_2^4)^{-1} .$$

This quaternion group is invariant under s_2, since

$$s_2^{-2}\,s_1^2\,s_2^5 = s_2\,s_1^2\,s_2^2,$$

and

$$s_1^2\,s_2^3 \,.\, s_2^{-1}\,s_1^2\,s_2^4 = s_1^2\,s_2^2\,s_1^2\,s_2^4 = s_1^2 \,.\, s_2^2\,s_1^{-1} \,.\, s_1^3\,s_2^4 = s_1^3\,s_2\,s_1^3\,s_2^4 = s_1^3\,s_2^2 \,.\, s_2^{-1}\,s_1^{-1} \,.\, s_2^4$$
$$= s_1^3\,s_2^2\,s_1\,s_2^2 = s_1^3 \,.\, s_2^2\,s_1^{-1} \,.\, s_1^2\,s_2^2 = s_2\,s_1^2\,s_2^2.$$

Hence $\{s_2^4,\ s_1^2 s_2^3,\ s_2^{-1} s_1^2 s_2^4\} \equiv$ the group of order 24 containing no subgroup of order 12. By adjoining the invariant operator s_2^3 to this G_{24}, we obtain a G_{48} whose properties are evident from these conditions. The given G_{24} is invariant under s_1, since

$$s_1\,s_2^3\,s_1 = s_1^2\,s_2^3, \quad s_1^{-1}\,s_2^{-1}\,s_1^2\,s_2^4\,s_1 = s_1^{-1}\,s_2^2\,s_1^2\,s_2\,s_1 = s_1^{-1} \,.\, s_2^2\,s_1^{-1} \,.\, s_1^2 \,.\, s_1\,s_2 \,.\, s_1 = s_2\,s_1^2\,s_2^2,$$
$$s_1^{-1}\,s_2^4\,s_1 = s_1^2\,s_2^3 \,.\, s_2^{-4} \,.\, s_2^6 = s_1^2\,s_2^{-1}\,s_1^{-1}\,s_1\,s_2^6 = s_1^3\,s_2^{-2}\,s_1\,s_2^6 = s_1^3\,s_2^4\,s_1.$$

The first two of these equations also complete the proof of the fact that the given quaternion group is invariant under G. As G_{24} is invariant under s_1, G_{48} must also have this property, and the latter contains s_1^2. Hence G is the group of order 96 obtained by adjoining s_1 to the given G_{48}. The quotient group of G with respect to G_{24} is cyclic. These results may be expressed as follows: *If two non-commutative operators satisfy the conditions* $s_1^4 = 1$, $s_1 s_2 = s_2^2 s_1^{-1}$, *they generate one of three groups of order* 24, 48 *and* 96 *respectively. The first of these is symmetric, and the other two have this group as a quotient group with respect to the invariant operators generated by* s_2^3. *If two commutative operators satisfy these conditions, they generate the cyclic group of order* 4.

The second and third sets of conditions are equivalent respectively to $s_1^4 = 1$, $(s_1 s_2)^2 = (s_2^{-1})^2$; $s_1^4 = 1$, $(s_1 s_2)^2 = (s_2^{-1})^3$. The latter of these has been investigated.* The former implies that the order of s_2 divides 16, since $(s_1 s_2^2)^4 = s_2^8$, s_2 being invariant under G. If s_2 is of order 16, the cyclic group of order 8 generated by $s_1 s_2^2$ has two operators in common with the cyclic group of order 16 generated by s_2, and each operator of the former is transformed into its inverse by a generator of the latter. Hence G is of order 64 when s_2 is of order 16, which is in accord with the general theorem stated in § 1, as $n = 4$. When s_2 is of order 8 and $s_1 s_2^2$ is of order 4, the cyclic groups generated by these operators have only the identity in common, since $(s_1 s_2^2)^2 = s_2^4$ implies that s_1 is of order 2. In this case G is therefore of order 32. If s_2 is of order 4 and $s_1 s_2^2$

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* *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 8.

is of the same order, G is the quaternion group when these operators have a common square, otherwise the order of G is 16. If the order of $s_1 s_2^2$ is 2, G is evidently abelian. Hence we have proved the theorem: *There are exactly four non-abelian groups which may be generated by two operators satisfying the conditions* $s_1^4 = 1$, $s_1 s_2 = s_2^{-3} s_1^{-1}$. *The orders of these groups are* 8, 16, 32 *and* 64 *respectively. The only abelian groups whose generators satisfy these conditions are the cyclic group of order* 8, *and the group of order* 16 *and of type* (3, 1).

§ 6. *Groups Defined by the Conditions* $s_1^5 = 1$, $s_1 s_2 = s_2^{a} s_1^{-1}$.

The following three sets are the only ones which require consideration:

$$s_1^5 = 1,\ s_1 s_2 = s_2^2 s_1^{-1};\quad s_1^5 = 1,\ s_1 s_2 = s_2^{-3} s_1^{-1};\quad s_1^5 = 1,\ s_1 s_2 = s_2^{-4} s_1^{-1}.$$

If the first set is satisfied, the order of s_2 divides 30, since

$$(s_1 s_2 s_1^{-1} s_2^{-1})^2 = (s_2^2 s_1^{-2} s_2^{-1})^2 = s_2^2 s_1^{-2} s_2 s_1^{-2} s_2^{-1} = s_2^2 s_1^2 s_2^2 s_1^2 s_2^{-1},$$

$$(s_2^2 s_1^{-2} s_2^{-1})^3 = s_2^2 s_1^2 s_2^2 s_1^2 s_2 s_1^{-2} s_2^{-1} = s_2^2 s_1^2 s_2^2 s_1 s_2^2 s_1^2 s_2^{-1} = s_2^5 s_1^2 s_2 s_1^{-1} s_2 s_1 s_2^2 s_1^{-1} s_2^{-2}$$
$$= s_2^8 s_1^2 s_2 s_1^3 s_2 s_1^{-1} s_2^{-2} = s_2^8 s_1^2 s_2 s_1^2 s_2^2 s_1^3 s_2^{-2},$$

$$(s_2^2 s_1^{-2} s_2^{-1})^5 = s_2^8 s_1^2 s_2 s_1^2 s_2^4 s_1^2 s_2^{-1} = s_2^{11} s_1 s_2^2 s_2^2 s_1 s_2^{-1} = s_2^{14} s_1 s_2 s_1 s_2^{-1} = s_2^{15}.$$

As the order of s_2 is a multiple of 3, when G is non-abelian, and a divisor of 30, it has one of the following values: 3, 6, 15, 30. If this value is 3, G is the icosahedral group. If the order of s_2 is 6, we can readily prove that G is a group of order 120, known as G_{120},* by letting

$$t_1 = s_1 s_2, \qquad t_2 = s_1^{-1} s_2^3,$$

and observing that

$$t_1^4 = 1, \qquad t_2^5 = t_1^2, \qquad (t_1 t_2)^3 = (t_2 t_1)^3 = s_2^{12} = 1.$$

When the order of s_2 is 15, it may be observed that

$$(s_1 s_2^{-3})^5 = 1, \qquad (s_2^{10})^3 = 1, \qquad (s_1 s_2^7)^2 = (s_1 s_2)^2 s_2^{12} = 1.$$

Hence $\{s_1 s_2^{-3},\ s_2^{10}\} \equiv$ the icosahedral group and G is the direct product of the cyclic group of order 5 and the icosahedral group. Finally, when the order of s_2 is 30, it is evident that

$$(s_1 s_2^7)^4 = 1, \qquad (s_1 s_2^7)^2 = (s_2^3 s_1^{-1})^5 = s_2^{15}, \qquad (s_2^3 s_1^{-1} . s_1 s_2^7)^3 = 1;$$

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* *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 10.

and hence $\{s_1 s_2^7, s_2^3 s_1^{-1}\} \equiv G_{120}$. Moreover, $\{s_1 s_2^7, s_2^3 s_1^{-1}, s_2^6\} \equiv G$, since it involves $s_1 s_2 = s_2^2 s_1^{-1}$, and hence also $s_2^2 s_1^{-1} \cdot s_1 s_2^{-3} = s_2^{-1}$. From this it results that G is the direct product of the cyclic group of order 5 and G_{120} when the order of s_2 is 30. This proves the following theorem: *If two non-commutative operators satisfy the conditions* $s_1^5 = 1$, $s_1 s_2 = s_2^2 s_1^{-1}$, *they generate one of the following four groups: the icosahedral group,* G_{120}, *the direct product of one of these groups and the cyclic group of order* 5. *If two commutative operators satisfy these conditions, they generate the cyclic group of order* 5.

As the third of the three given pairs of conditions is equivalent to $t_1^2 = t_2^3$, $(t_1 t_2)^5 = 1$, the groups which may be generated by two operators satisfying these conditions are known.* It remains therefore only to consider the possible groups when

$$s_1^5 = 1, \qquad (s_1 s_2)^2 = s_2^{-2}.$$

In this case the order of s_2 divides 20, since $(s_1 s_2^2)^5 = s_2^{10}$, and hence s_2^{10} is both invariant under $s_1 s_2$ and also tranformed into its inverse by this operator. If the order of s_2 is 20, the order of $s_1 s_2^2$ is 10, since G is not abelian. The order of G is therefore 100 when s_2 is of order 20. If the order of s_2 is 10, that of $s_1 s_2^2$ must be 5, and G is the group of order 50, which may be obtained by extending the group of order 25 and of type (1, 1) by means of an operator of order 2 which is commutative with one of the two independent generators of this group, but transforms the other into its inverse. If the order of s_2 is 4, that of $s_1 s_2^2$ is 10, and G is the associate-dihedral group of order 20, while G is the dihedral group of order 10 when s_2 is of order 2. As we have considered all the possible values of the order of s_2 that may give rise to a non-abelian group, there results the theorem: *There are exactly four non-abelian groups which may be generated by two operators satisfying the conditions* $s_1^5 = 1$, $s_1 s_2 = s_2^{-3} s_1^{-1}$. *The orders of these groups are* 100, 50, 20, 10 *respectively. The abelian groups which may be generated by two such operators are cyclic and of orders* 5, 10 *and* 20 *respectively.*

§ 7. *Groups Defined by the Conditions* $s_1^n = 1$, $s_1 s_2 = s_2^{-2} s_1^{-2}$; $s_1^n = 1$, $s_1 s_2 = s_2^2 s_1^2$.

If two operators satisfy the condition $s_1 s_2 = s_2^{-2} s_1^{-2}$, they are either of the same order or the order of one of them is three times that of the other.

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* *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 12.

Moreover, the operator $s_1 s_2$ generates a cyclic group of odd order which is invariant under G, and the following relations exist for all values of m:

$$(s_1 s_2)^{\frac{2}{3}4^m+\frac{1}{3}} = s_1^{2m+1} s_2^{2m+1}, \qquad (s_2 s_1)^{\frac{2}{3}(4^m-1)} = s_1^{2m} s_2^{2m}.^*$$

From these conditions it results that the order of G is finite for any finite value of n in the conditions $s_1^n = 1$, $s_1 s_2 = s_2^{-2} s_1^{-2}$. As G is also a solvable group of a very elementary type, it is easy to determine the possible groups for a given value of n. For instance, when $n = 2$, G is the cyclic group of order 6 or the group of order 2; when $n = 3$, G can be non-abelian only when it is of order 27 and involves operators of order 9. If G is abelian when $n = 3$, it is either of order 3 or it is the non-cyclic group of order 9. Hence the theorem: *If two generators of a group satisfy the conditions* $s_1^3 = 1$, $s_1 s_2 = s_2^{-2} s_1^{-2}$, *the group is one of the following three: the non-abelian group of order* 27 *which contains operators of order* 9, *the non-cyclic group of order* 9, *or the group of order* 3.

The merits of this theorem consist mostly in serving as a type of theorems which may be established for the different values of n. The fact that deserves emphasis in this connection is that for every given value of n the number of the possible groups is finite, and that these groups have such elementary properties as to make a complete determination of the possible groups for a given value of n an easy matter. The conditions

$$s_1^n = 1, \qquad s_1 s_2 = s_2^2 s_1^2$$

present much greater difficulties. These have been solved only for a few values of n. The latter condition implies that the orders of s_1, s_2 are either the same or that the order of one of these operators is odd and that of the other is the double of this odd number. When $n = 3$, G is the alternating group of order 12, the group of order 24 which does not contain a subgroup of order 12, or the group of order 3. When $n = 4$, G is either the cyclic group of order 4 or the holomorph of the group of order 5.†

§ 8. *Conclusion.*

The pairs of conditions which have been completely investigated, in the present paper, as regards whether they may be satisfied by two generators of only a finite number of groups, are as follows: $s_1^n = 1$, $s_2^m = 1$; $s_1^n = 1$, $s_1^\alpha = s_2^\beta$; $(s_1 s_2)^n = 1$, $s_1^2 = s_2^2$; $s_1^n = 1$, $s_1 s_2 = s_2^\alpha s_1^\beta$, where either α or β is ± 1; $s_1^n = 1$, $s_1 s_2 = s_2^{-2} s_1^{-2}$. A few pairs of conditions of the form $s_1^n = 1$, $s_1 s_2 = s_2^2 s_1^2$ have also been considered. The latter of these conditions was partially investigated by Cayley as early as 1878, who observed that all the operators of a non-abelian group generated by s_1, s_2 can not be written in the form $s_1^\alpha s_2^\beta$. Among pairs of conditions which have been examined previously are those in which each one of two generating operators is transformed into a power of itself by the other. The fact that the two equivalent pairs

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* *Bulletin of the American Mathematical Society*, Vol. XV (1909), p. 162.

† Netto, *Crelle*, Vol. CXXVIII (1905), p. 255.

$$s_1^{-1} s_2 s_1 = s_2^3,\ s_2^{-1} s_1 s_2 = s_1^3; \qquad s_2^{-1} s_1 s_2 = s_1^{-1},\ s_1^{-1} s_2 s_1 = s_2^{-1}$$

are satisfied only by the generators of the four group and those of the quaternion group, has been especially emphasized.

As a negative result it may be added that an abelian group of finite order can not be defined by two conditions when one of them is of the common form $s_1 s_2 = s_2 s_1$. This follows directly from the proof given in § 1 of the fact that a single condition between two or more operators is insufficient to fix an upper limit for the order of the group generated by these operators. Two conditions of the form $s_1^{-1} s_2 s_1 = s_2^\alpha$, $s_2^{-1} s_1 s_2 = s_1^\beta$ may, however, be sufficient to restrict G to abelian groups whose orders are less than a given number. In fact, it is only necessary to select α, β in such a way that $\alpha - 1$, $\beta - 1$ are relatively prime, in order to attain this result.* If, moreover, $\alpha - 1$ and $\beta - 1$ are prime numbers, G is uniquely defined by these conditions, and only then.

When the two operators s_1, s_2 satisfy two conditions, these conditions may be sufficient to restrict G to a particular group, which may be abelian or non-abelian; for instance, the conditions $s_1^5 = 1$, $s_1^2 = s_2^5$ are satisfied only by two generators of the cyclic group of order 25, while the conditions $s_1^2 = 1$, $s_1 s_2 = s_2^2 s_1$ define the symmetric group of order 6. On the other hand, these two conditions may restrict G to a finite number of groups (varying from 2 to a number dependent upon an exponent), as is the case in most of the pairs of conditions examined above. Finally, the two conditions may not restrict G to a finite number of groups, but they may be sufficient to determine some fundamental properties of G.

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* *Quarterly Journal of Mathematics*, Vol. XXXVII (1906), p. 287.

It is of interest to observe that when the two conditions are sufficient to restrict G to a particular abelian group, this group is necessarily cyclic. In fact, these conditions may be reduced to the form $s_1^\alpha = s_2^\beta$, $s_1^\gamma = s_2^\delta$. If one of the exponents α, β, γ, δ is zero, the possible groups were considered above. If this is not the case, it is necessary that the numbers α, β are relatively prime, as G would evidently not be uniquely determined if they had a common factor. The fact, that α, β are relatively prime implies that s_1 is generated by s_2 and hence G is cyclic. Hence the theorem: *If two conditions imposed upon s_1, s_2 restrict G to a particular abelian group, this group must be cyclic.* From this theorem it results that a non-cyclic abelian group can not be defined uniquely by less than three conditions. It is evident that some such groups may be defined by three conditions.